LECTURE

5

LIMITS, CONTINUITY AND DIFFERENTIATION OF REAL FUNCTIONS OF ONE VARIABLE

Limits of functions

Definition 5.1 Let A ⊆ R be a set. An element c ∈ R ..= R ∪ {−∞,+∞} is said to be a cluster point (or accumulation point) of A if

∀V ∈ V(c), V ∩ (A \ {c}) = ∅.

The set of all accumulation points of A is denoted by

A ..= {c ∈ R | ∀V ∈ V(c), V ∩ A \ {c} = ∅}

and it is called the derived set of A. The elements of

A \ A = {a ∈ A | ∃V ∈ V(a) s.t. V ∩ A = {a}}

are called isolated points of A.

Theorem 5.2 (Sequential characterization of accumulation points) Let A ⊆ R be a set. For any c ∈ R the following assertions are equivalent:

1◦ c is an accumulation point of A, i.e., c ∈ A . 2◦ there exists a sequence (xn) in A \ {c} such that lim n→∞xn = c.

Proof. 2◦ ⇒ 1◦: Assume that (xn) is a sequence in A\{c} such that lim n→∞xn = c. Then, for any V ∈ V(a), ∃nV ∈ N such that xn ∈ V for all n ∈ N, n ≥ nV. In particular, we have xnV ∈ V ∩ (A \ {c}), hence V ∩ (A \ {c}) = ∅. It follows that c ∈ A .

1◦ ⇒ 2◦: Assume that c ∈ A . We distinguish three cases. Case 1: c ∈ R. In this case, for every n ∈ N we have (c − 1n,c + 1n) ∈ V(c), hence there exists xn ∈ (c − 1n,c + 1n) ∩ (A \ {c}). The sequence (xn) converges to c, since |xn − c| < 1n for all n ∈ N.

Case 2: c = −∞. In this case, for every n ∈ N we have (−∞,−n) ∈ V(c), hence there exists xn ∈ (−∞,−n) ∩ (A \ {c})=(−∞,−n) ∩ A. Since xn < −n for all n ∈ N, we infer that lim n→∞xn = −∞ = c.

Case 3: c = +∞. For every n ∈ N we have (n,∞) ∈ V(c), hence there exists xn ∈ (n,∞) ∩ (A \ {c}). Since xn > n for all n ∈ N, we infer that lim n→∞xn = +∞ = c. D

1

Example 5.3 a) Let A = {0,1}. We have A = ∅, hence 0 and 1 are isolated points of A. Notice that finite sets have no accumulation points! b) Let A = (0,1). Then A = [0,1], hence A has no isolated points. Notice that 0 and 1 are accumulation points of A, but do not belong to A. c) Let A = (0,1) ∪ (1,2), In this case we have A = [0,2]. The set A has no isolated points. d) Let A = N. Then A = {+∞} and all points of A are isolated. e) Let A = [0,1]∩Q. By the density of Q in R, it follows that A = [0,1]. The set A has no isolated points.

f) Let A =

{1n }| n ∈ N. In this case, A = {0} and all points of A are isolated.

Definition 5.4 Let f : A → R be a function, defined on a nonempty set A ⊆ R, and let c ∈ A . We say that f has a limit at c if there exists l ∈ R such that

∀V ∈ V(l), ∃U ∈ V(c) s.t. f(x) ∈ V, ∀x ∈ U ∩ (A \ {c}). (5.1)

Remark 5.5 If f has a limit at c, then there exists a unique l ∈ R satisfying (5.1). In this case l is called the limit of f at c and we write

x→clim f(x) = l or

f(x) → l as x → c,

which reads as follows: “f has the limit l at c” or “f(x) approaches L as x approaches c”.

Proof (of the uniqueness of l): Suppose by the contrary that there exist l1,l2 ∈ R, l1 = l2 such that (5.1) holds for l = l1 as well as for l = l2. Choose V1 ∈ V(l1) and V2 ∈ V(l2) such that V1∩V2 = ∅. Then, on the one hand, there exists U1 ∈ V(c) such that f(x) ∈ V1 for all x ∈ U1 ∩(A \ {c}). On the other hand, there exists U2 ∈ V(c) such that f(x) ∈ V2 for all x ∈ U2∩(A \ {c}). Since U1∩U2 ∈ V(c) and c ∈ A , we can find some x0 ∈ (U1 ∩ U2)∩(A \ {c}). This yields f(x0) ∈ V1∩V2, a contradiction. DRemark 5.6 f has no limit at c if ∀l ∈ R, ∃V ∈ V(l), ∀U ∈ V(c), ∃x ∈ U ∩ (A \ {c}), f(x) /∈ V .

Theorem 5.7 (Characterization of limits in terms of ε-δ) Let f : A → R be a function de- fined on a nonempty set A ⊆ R and let c ∈ A .

1◦ If c ∈ R, then 2◦ (i) (ii) (iii) If c x→clim x→clim x→clim = f(x) −∞, f(x) f(x)=+∞⇔∀ε = = then

l −∞ ∈ R ⇔ ⇔ ∀ε ∀ε > > > 0, 0, 0, ∃δ ∃δ ∃δ > > > 0, 0, 0, ∀x ∀x ∀x ∈ ∈ ∈ A, A, A, 0 0 0 < < < |x |x |x − − − c| c| c| < < < δ, δ, δ, f(x) |f(x) f(x) < > − −ε. ε. l| < ε.

(i) x→−∞lim f(x) = l ∈ R ⇔ ∀ε > 0, ∃δ > 0, ∀x ∈ A, x < −δ, |f(x) − l| < ε. (ii) x→−∞lim f(x) = −∞ ⇔ ∀ε > 0, ∃δ > 0, ∀x ∈ A, x < −δ, f(x) < −ε. 3◦ (iii) If c x→−∞= lim ∞, f(x)=+∞⇔∀ε then

> 0, ∃δ > 0, ∀x ∈ A, x < −δ, f(x) > ε.

(i) (ii) (iii) x→∞lim x→∞lim x→∞lim f(x) f(x) f(x)=+∞⇔∀ε = = l −∞ ∈ R ⇔ ⇔ ∀ε ∀ε > > > 0, 0, 0, ∃δ ∃δ ∃δ > > > 0, 0, 0, ∀x ∀x ∀x ∈ ∈ ∈ A, A, A, x>δ, x>δ, x>δ, f(x) |f(x) f(x) < > − −ε. ε. l| < ε.

2

Theorem 5.8 (Heine’s sequential characterization of limits) Let f : A → R be a function defined on a nonempty set A ⊆ R and let c ∈ A . For any l ∈ R the following assertions are equivalent:

1◦ 2◦ x→cfor lim every f(x) = sequence l.

(xn) in A \ {c} with n→∞lim xn = c we have n→∞lim f(xn) = l.

Example 5.9 Function f : R∗ → R, defined two sequences (xn) and (x n) in R∗, xn ..= 12πn by and f(x) x n .= ...= sin 2πn x11

, + has π2 no limit at 0. Indeed, there exist , such that n→∞lim xn = n→∞lim x n = 0, but n→∞lim f(xn)=0 = n→∞lim f(x n)=1.

Remark 5.10 Function f : A → R has a limit at c ∈ A if and only if for every sequence (xn) in A \ {c} with n→∞lim xn = c the sequence (f(xn)) has a limit in R.

Example 5.11 The “sign” function sgn : R → R, defined by sgn(x) .=

.−1, if x < 0, 0, 1, if x = 0, if x > 0

has no

limit at c .= .0. Indeed, defining the sequence (xn) in R \ {0} by xn .= .but the sequence (sgn(xn)), i.e., ((−1)n), has no limit.

(−1)n n

, we have n→∞lim xn = 0,

By means of Heine’s Theorem 5.8 we can derive several important results for limits of functions from corresponding ones known for limits of sequences. To illustrate this, we present a few results.

Theorem 5.12 Let f,g : A → R be two functions defined on a nonempty set A ⊆ R and let c ∈ A . Suppose that there is U ∈ V(c) such that f(x) ≤ g(x), ∀x ∈ U ∩ (A \ {c}). (i) If (ii) If (iii) If x→cx→cx→clim lim lim f(x) f(x)=+∞, g(x) = and −∞, x→clim then then g(x) x→clim x→clim exist, f(x) g(x)=+∞. then = −∞.

x→clim f(x) ≤ x→clim g(x).

Remark x + x 1

> 1,∀x 5.13 > Strict 0, but inequalities x→∞

lim x + x are not preserved under the limiting process. For instance, we have 1

= 1.

Theorem 5.14 (Squeeze Theorem for functions) Let f,g,h : A → R be three functions defined on a nonempty set A ⊆ R and let c ∈ A . If there is U ∈ V(c) such that f(x) ≤ g(x) ≤ h(x), ∀x ∈ U ∩ (A \ {c}) and x→clim f(x) = x→clim h(x) = l ∈ R, then limx→c g(x) = l.

Proof. Let (xn) be a sequence g(xn) ≤ h(xn), ∀n ≥ n0. On in the A \ {c} other such hand, that by n→∞lim Theorem xn Applying the Squeeze Theorem for sequences, it follows that 5.8, we obtain that x→clim g(x) = l. = c. Thus, ∃n0 ∈ N such that f(xn→∞lim 5.8, g(xn→∞lim n) = f(xl. n) Using = n→∞lim again h(xTheorem n) n) ≤ = l.

D

Theorem 5.15 Let f : A → R be a function defined on a nonempty set A ⊆ R and let c ∈ A . For any l ∈ R we have x→clim f(x) = l ⇐⇒ x→clim |f(x) − l| = 0.

Definition 5.16 Let f : A → R be a function defined on a nonempty set A ⊆ R and let c ∈ R.

If c is an accumulation point of A ∩ (−∞,c) and the restriction f|A∩(−∞,c) has a limit at c, then we call this limit the left-hand limit of f at c and we denote it by

x→c lim x<c f(x) or lim f(x) or x→c− x↗clim f(x).

3

Similarly, if c is an accumulation point of A ∩ (c,∞) and f|A∩(,∞) has a limit at c, then we call this limit the right-hand limit of f at c and we denote it by

lim x→c x>c f(x) or lim x→c+ f(x) or lim x↘cf(x).

These two limits are called one-sided limits of f at c.

Theorem 5.17 (Characterization of limits using one-sided limits) Let f : A → R be a func- tion defined on a nonempty set A ⊆ R, let l ∈ R and let c ∈ R be an accumulation point of both the sets A ∩ (−∞,c) and A ∩ (c,+∞). Then

lim x→cf(x) = l ⇐⇒ lim x→c x<c f(x) = lim x→c x>c f(x) = l.

Remark 5.18 When c is an accumulation point of both the sets A ∩ (−∞,c) and A ∩ (c,+∞), the usual limit lim x→cf(x) is also called the two-sided limit of f at c.

Continuous functions

Definition 5.19 Let A ⊆ R, f : A → R, c ∈ A. We say that f is continuous at c if

∀V ∈ V(f(c)),∃U ∈ V(c) such that ∀x ∈ U ∩ A we have f(x) ∈ V.

In this case we call c a continuity point of f.

If f fails to be continuous at c, then we say that f is discontinuous at c and that c is a disconti- nuity point of f.

If B is a subset of A, we say that f is continuous on B if it is continuous at every point of B. In particular, if f is continuous on A, we simply say that f is continuous.

Remark 5.20 (i) An important difference between the notions of limit and continuity is that the point c is now assumed to belong to A (but not necessarily to A ) so that f(c) makes sense. (ii) If c ∈ A ∩ A , then f is continuous at c if and only if lim x→cf(x) = f(c). (iii) If c is an isolated point of A, then ∃U ∈ V(c) such that U ∩ A = {c}. Thus, f is continuous at c.Theorem 5.21 (Characterizations of continuity) Let A ⊆ R, f : A → R, c ∈ A. The following assertions are equivalent:

1◦ f is continuous at c. 1◦ ∀ε > 0, ∃δ > 0, ∀x ∈ A, |x − c| < δ, |f(x) − f(c)| < ε. 1◦ For every sequence (xn) in A with lim n→∞xn = c we have lim n→∞f(xn) = f(c).

Remark 5.22 Sums, products, quotients and compositions of continuous functions (when defined) are continuous.

Definition 5.23 Let A ⊆ R, f : A → R, let c ∈ A be a discontinuity point of f. We say that c is a discontinuity point of the first kind of f (or that f has a discontinuity of the first kind at c) if the one-sided limits of f at c both exist and are finite. The discontinuities that are not of the first kind are called discontinuities of the second kind.

Remark 5.24 If c is a discontinuity of the first kind, then it is either a jump discontinuity when the one-sided limits are distinct or a removable discontinuity if the one-sided limits coincide (but they are not equal to f(c)).

4

Example 5.25 (i) The function f : R → R,

f(x) =

{sin x1, if x = 0, 0, if x = 0

has a discontinuity of the second kind at 0. (ii) The function sgn(x) has a jump discontinuity at 0. (iii) The function f : R → R,

f(x) =

{xsin x1, if x = 0, 1, if x = 0

has a removable discontinuity at 0.

Definition 5.26 Let A ⊆ R. A function f : A → R is said to be bounded on A if the set f(A) is bounded, i.e., there exists M > 0 such that ∀x ∈ A, |f(x)| ≤ M. We say that f attains its maximum if there exists x ∈ A such that ∀x ∈ A, f(x) ≤ f(x). Likewise, we say that f attains its minimum if there exists x ∈ A such that ∀x ∈ A, f(x) ≤ f(x). In this case x is called a maximum point for f and x is called a minimum point for f.

Theorem 5.27 (Weierstrass’ theorem on extrema of continuous functions) Let a, b ∈ R with a<b and let f : [a, b] → R be continuous on [a, b]. Then f is bounded and it attains both its maximum and minimum on [a, b].

Remark 5.28 (i) The function f can be unbounded if

• the interval is unbounded: f : [0,+∞) → R, f(x) = x.

• the interval is not closed: f : (0,1] → R, f(x)=1/x.

• f is not continuous: f : [0,1] → R, f(x) =

{1/x, if x ∈ (0,1],

0, if x = 0. (ii) A maximum (minimum) point is not necessarily unique.

Theorem 5.29 (Intermediate Value Theorem, Bolzano-Darboux) Let f : [a, b] → R be a continuous function, where a, b ∈ R, a<b, and let v ∈ R. If min{f(a),f(b)} <v< max{f(a),f(b)}, then there exists (at least one) c ∈ (a, b) such that f(c) = y.

Remark 5.30 (i) Location of Roots (Bolzano Theorem): If f : [a, b] → R is continuous and f(a) · f(b) < 0, then ∃c ∈ (a, b) such that f(c)=0. (ii) If I is an interval and f : I → R is continuous, then for every interval J ⊆ I the set f(J) is an interval (possibly degenerated into a singleton). (iii) If f : [a, b] → R is continuous, then f ([a, b]) is a compact interval (possibly degenerated into a singleton).

Differentiation of functions

Definition 5.31 Let A ⊆ R and c ∈ A∩ A . A function f : A → R has a derivative at c if the limit

x→c lim f(x) x − − f(c)

c

exists (in R). In this case, the above limit is called the derivative of f at c and is denoted by

f (c).

If f has a finite derivative at c, then f is said to be differentiable at c.

If B is a subset of A, we say that f is differentiable on B if it is differentiable at every point of B. In this case, the function f : B → R, x ∈ B ↦→ f (x) ∈ R is called the derivative of f on B. In particular, if f is differentiable on A, then we simply say that f is differentiable.

5

Example 5.32 Let f : R → R, f(x) = x2. At any c ∈ R,

f (c) = x→c

lim f(x) x − − f(c)

c = x→c

lim xx 2 − − c2

c = x→clim (x + c)=2c.

Thus, f is differentiable on R and ∀x ∈ R, f (x)=2x.

Theorem 5.33 Let A ⊆ R, f : A → R and c ∈ A ∩ A . If f is differentiable at c, then f is also continuous at c.

Proof. The conclusion follows byf(x) − f(c) = f(x) x − − f(c)

c (x − c)

for all x ∈ A, x = c. D

Remark 5.34 1) A function can have a derivative at a point without being continuous at that point: sgn(x) has a derivative at 0, sgn (0) = +∞, but it is not continuous at 0.

2) A function can be continuous at some point without being differentiable at that point: f(x) = |x| is continuous at every x ∈ R, but has no derivative at 0.

3) There exist functions that are continuous on R, but nowhere-differentiable. For instance, let d(x) be the distance from x ∈ R to the integer closest to x, that is, d(x) = |x| for x ∈ [−1/2,1/2], extended on R by periodicity. Then, the function f : R → R, defined by

f(x) .=

.∑∞n=0

d(2n2n x)

, ∀x ∈ R

is continuous on R, but nowhere differentiable.

Definition 5.35 Let f : A → R be a function defined on a nonempty set A ⊆ R and let c ∈ A. If c is an accumulation point of A ∩ (−∞,c), then we say that f has a left-hand derivative at c if the following left-hand limit exists

f l(c) .= .x→c lim x<c

f(x) − f(c)

x − c ∈ R.

In this case, derivative at f c), l(c) then is called f the left-hand derivative of f at c. is said to be left-hand differentiable If at f c.

l(c) ∈ R (i.e., f has a finite left-hand

Similarly, when c is an accumulation point of A∩(c,∞), we say that f has a right-hand derivative at c if the following right-hand limit exists

f r(c) .= .x→c lim x>c

f(x) − f(c)

x − c ∈ R

and we say that f is right-hand differentiable at c whenever f r(c) ∈ R.

Remark 5.36 If A = [a, b], where a, b ∈ R with a<b, then the differentiability of f : A → R at a is actually the right-hand differentiability of f at a while the differentiability of f at b is actually the left-hand differentiability of f at b.

6

Theorem 5.37 (Calculus Rules) Let f,g : A → R be two functions, defined on a nonempty set A ⊆ R, that are differentiable at c ∈ A ∩ A .

1◦ If α ∈ R, then function αf is differentiable at c and

(αf) (c) = αf (c).

2◦ Function f + g is differentiable at c and

(f + g) (c) = f (c) + g (c).

3◦ Function fg is differentiable at c and

(fg) (c) = f (c)g(c) + f(c)g (c).

4◦ If g(c) = 0, then function f/g (defined on some neighborhood of c) is differentiable at c and

(fg)

(c) = f (c)g(c) − f(c)g (c)

(g(c))2 .

Theorem 5.38 (Chain rule) Let I,J ⊆ R be intervals, c ∈ I, f : I → R and g : J → R, such that f(I) ⊆ J. If f is differentiable at c and g is differentiable at f(c), then g ◦f : I → R is differentiable at c and

(g ◦ f) (c) = g (f(c))f (c).

Theorem 5.39 (Inverse Function Theorem) Let I,J ⊆ R be intervals, c ∈ I and let f : I → J be an invertible function (i.e., f is a bijection). If f is differentiable at c, f (c) = 0 and f−1 : J → I is continuous at f(c), then f −1 is differentiable at f(c) and

(f −1) (f(c)) = 1

f (c).

Local extrema and derivatives

Definition 5.40 Let A ⊆ R and f : A → R. We say that f attains a local maximum (local minimum) at c ∈ A if there exists V ∈ V(c) such that c is a maximum point (minimum point) for f|A∩V . We say that f attains a local extremum at c ∈ A if it attains either a local maximum or a local minimum at c.

Theorem 5.41 (Fermat) Let a, b ∈ R, a<b and f : (a, b) → R. If c ∈ (a, b), f has a derivative at c and f attains a local extremum at c, then f (c)=0.

Remark 5.42 (i) The above result may fail if one does not assume that f has a derivative at c (take f : (−1,1) → R, f(x) = |x|, c = 0) or if the open interval is replaced by a closed one (take f : [0,1] → R, f(x) = x. Then f attains a minimum at c = 0, but f (0) = 1). (ii) If f (c) = 0, it does not follow that f attains a local extremum at c (take f : (−1,1) → R, f(x) = x3, c = 0).

Theorem 5.43 (Darboux) Let a, b ∈ R, a<b, and let f : (a, b) → R be differentiable. Then, for any x1,x2 ∈ (a, b) and v ∈ R such that

x1 < x2 and min{f (x1),f (x2)} <y< max{f (x1),f (x2)},

there exists x ∈ (x1,x2) such that f (x) = y.

7

Remark 5.44 The derivative of a differentiable function is not always continuous. For instance, consider the function f : R → R,

f(x) =

{x2 sin x1, if x = 0, 0, if x = 0.

Then f is differentiable on R, but f is not continuous at 0.

Definition 5.45 A function f : I → R, defined on an interval I ⊆ R, is called continuously differentiable if it is differentiable and its derivative f : I → R is continuous.

Theorem 5.46 (Rolle) Let a, b ∈ R, a < b and f : [a, b] → R. If f is continuous on [a, b], differentiable on (a, b) and f(a) = f(b), then there exists c ∈ (a, b) such that f (c)=0.

Theorem 5.47 (Lagrange’s Mean Value Theorem) Let a, b ∈ R, a<b and f : [a, b] → R. If f is continuous on [a, b] and differentiable on (a, b), then there exists c ∈ (a, b) such that

f(b) − f(a) = f (c)(b − a).

Theorem 5.48 (Cauchy’s Generalized Mean Value Theorem) Let a, b ∈ R, a<b. If f,g : [a, b] → R are continuous on [a, b] and differentiable on (a, b), then there is c ∈ (a, b) such that

(f(b) − f(a))g (c)=(g(b) − g(a))f (c).

Theorem 5.49 (L’Hôpital’s rule for right-hand limits) Let a, b ∈ R with a<b and let f,g : (a, b) → R be two functions satisfying the following conditions:

1) f and g are differentiable on (a, b). 2) ∀x ∈ (a, b), g (x) = 0. 3) x→a lim x>a f(x) = x→a lim x>a g(x) ∈ {−∞,0,+∞}.

4) x→a lim x>a

f g (x) (x)

= l ∈ R.

Then x→a lim x>a

f(x) g(x) = l.

Theorem 5.50 (L’Hôpital’s rule for left-hand limits) Let a, b ∈ R with a < b and let f,g : (a, b) → R be two functions satisfying the following conditions:

1) f and g are differentiable on (a, b). 2) ∀x ∈ (a, b), g (x) = 0. 3) x→b lim x<b

f(x) = x→b lim x<b

g(x) ∈ {−∞,0,+∞}.

4) x→b lim x<b

f g (x) (x)

= l ∈ R.

Then x→b lim x<b

f(x) g(x) = l.

Theorem 5.51 (L’Hôpital’s rule for two-sided limits) Let a, b ∈ R with a < b and let c ∈ (a, b). Let f,g : (a, b) → R be two functions satisfying the following conditions:

1) f and g are differentiable on (a, b) \ {c}. 2) g (x) = 0 for all x ∈ (a, b) \ {c}. 3) 4) x→clim x→c

lim f(x) f g (x) (x)

= = x→clim l ∈ g(x) R.

∈ {−∞,0,+∞}.

Then x→c

lim f(x)

g(x) = l.

8

sinx Example 1 − cosx

x2 = x→0

lim 2x = x→0

lim cosx

(ii) (iii) x→0 lim x→∞ lim ex x − 5.52 1

= (i) x→0

lim x→0

lim 1/xe1 x1 2 = 12.

= 1.

= 0.

(iv) x→0 lim x>0

lnxx = x→∞

lim (sinxx · 1 cosxlnx ln(sinx) = x→0 lim x>0

(cosx)/(sinx) 1/x

= x→0 lim x>0

)

= 1.

9